Morphisms of Representations

$\varphi: G \to GL(V)$, $\psi: G \to GL(W)$

a morphism of representations from φ to ψ is a linear map $T: V \to W$ such that

$$\psi_g \circ T = T \circ \varphi_g \quad \forall\ g \in G$$

(also intertwining operator)

<u>Note</u>: If T iso of vector spaces, $T^{-1}: W \to V$

$$\Rightarrow \quad \psi_g = T \circ \varphi_g \circ T^{-1} \quad (\Leftrightarrow \text{equivalence})$$

V, W vector spaces

$$\underbrace{\mathrm{Hom}(V,W)}_{\text{is a vector space}} := \{T: V \to W \text{ linear maps}\}$$

$c_1 T_1 + c_2 T_2$

$\varphi: G \to GL(V)$, $\psi: G \to GL(W)$

Define

$$\mathrm{Hom}_G(\varphi, \psi) := \{ T \in \mathrm{Hom}(V,W) \mid \psi_g T = T\varphi_g \ \forall g \in G \}$$

vector subspace of $\mathrm{Hom}(V,W)$:

if $T_1, T_2 \in \mathrm{Hom}_G(\varphi,\psi)$

then $\underline{c_1T_1 + c_2T_2} \in \mathrm{Hom}_G(\varphi,\psi)$, $c_1,c_2 \in \mathbb{C}$

Prop: $T: V \to W$ is a morphism of representations from φ to ψ

$[\varphi: G \to GL(V), \ \psi: G \to GL(W)]$

then $\mathrm{Ker}(T) \leq V$ and $T(V) \leq W$ are invariant subspaces

Proof: if $v \in \mathrm{ker}(T)$, $g \in G$

$T(\varphi_g(v)) = \psi_g(T(v)) = \psi_g(0) = 0$ so

$\varphi_g(\mathrm{Ker}(T)) \subseteq \mathrm{Ker}(T)$, also applies to $\varphi_{g^{-1}}$